Fly By Black Hole Physics*

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In this article we will examine several aspects of black holes with the use of fly by night technique. We are primarily interested in coming to the same conclusion, at least the same proportionality, without a hard derivation such as Hawking's original paper on black hole radiation. We will cover Schwarzschild radius, Hawking temperature, black hole entropy in the black hole section. Then, we will examine some systems that could approximate a black hole and compare them. We will present some currently open questions with attempts to explain them. Finally, we will end with methods of extracting energy from black holes.

I. INTRODUCTION

The black hole is a mysterious entity of enormous mass and energy. In fact, black holes will probably be the last celestial bodies present near the very end of the universe. Black holes display some of the very bizarre properties: for example, its entropy is not an extensive quantity as usual anymore, but rather is proportional to surface area. Understanding black holes will possibly deepen our understanding about information theory, modify our formulation of gravity, and perhaps even lead to the construction of quantum gravity. In April 10, 2019, the world was shocked to see the first picture of supermassive black hole at the center of Messier 87 revealed. It is even speculated that supermassive black holes are necessary for the existence for a galaxy. Therefore, it is of great importance to strive to learn about black hole. Actually, there are plenty of things we can do with just fly by night physics, even to such a peculiar entity.

II. ALL ABOUT BLACK HOLES

In this section we are going to talk about some general properties of black holes and some unsolved questions.

First of all, wee need to clarify that what we are examining in this section is merely the simplest form of a black hole: the Schwarzschild metric is a solution for a point mass without electric charge and angular momentum. Four types of solutions have been found when solving general relativity, and they are listed in Table I.

If charges were to involve, then Schwarzschild radius would vary due to the repulsion between the same charges. In fact, we can say the net charges of natural black holes are very very small because otherwise it would take a lot of matter to fight through electric repulsion. If angular momentum were to involve, they would cancel partially with gravity and also causing black holes slightly harder to form. The spinning black holes also has an effect called frame-dragging, and we can cleverly extract energy from the black hole with either Blandford–Znajek process or Penrose Mechanism. We will discuss this with more detail in a later section. We are usually justified to ignore net charge, but not angular momentum so the later results would have to be mod-

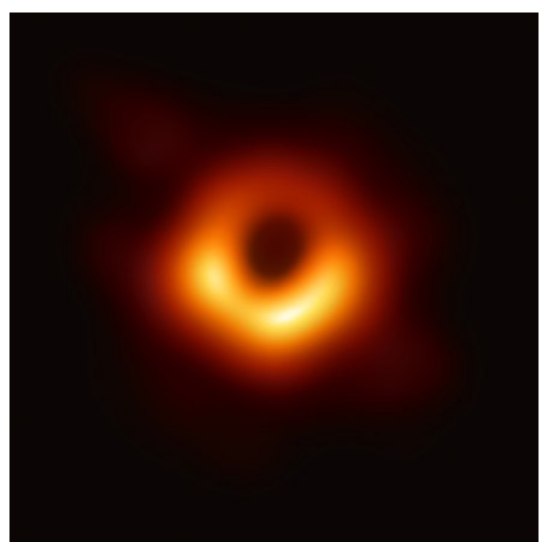

FIG. 1. The first image of a black hole captured by the Event Horizon Telescope.

* Since we want to know what happens inside a black hole and the contents inside the event horizon is not visible, it is essentially "night".

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ified for a real black hole, which is usually spinning at rate of about 1000 times per second.

A. Schwarzschild Radius

Imagine a star who is “debt”, that is, even its whole mass is converted into energy, that is not enough to pay the debt caused by gravitation potential. Namely:

$$\frac{GM}{R^2} \sim Mc^2 \tag{1}$$

Therefore, we can guess that when this happens, the star turns into something strange. Actually, this radius is the Schwarzschild radius (denoted by r_s) , if we do the complete math and add in a factor of two:

$$r_s \sim \frac{GM}{c^2} \tag{2}$$

The interpretation of this quantity is, if anything falls within this radius, it can never pay off its “debt”, meaning that this object can never return. For this reason this is also called the event horizon radius.

B. Hawking Temperature

Surprisingly, the hawking temperature can also be reasoned using a little bit of facts. First of all, we know that black holes exist and they are supermassive objects. Second of all, we cannot directly see them from the sky. What this means is that the Hawking temperature of a black hole must be inversely proportional to some powers of its mass, otherwise blackholes would probably be the only celestial entity we see in the sky, dwarfing any other star in the sky except probably a supernova close to the earth. We almost never see odd powers of mass, so it is a good start to guess it is one. Also, invoking that mass never appears alone, G must accompany it. Now if we use natural units, we know that $\frac{GM}{R}$ has units of energy per length, and with $h = c = 1$, we can see that GM has units of inverse length. We know that temperature has units of energy, aka units of mass, therefore we get:

$$T \sim \frac{1}{M} \tag{3}$$

Note that it is very tempting to directly set $T \sim \frac{GM}{R}$ since both have units of energy. However, we eliminated this possibility because, again, we cannot see black holes in the sky.

TABLE I. Four types of black holes

Schwarzschild	Kerr	Reissner–Nordström	Kerr–Newman
J = 0	J $\neq$ 0	J = 0	J $\neq$ 0
Q = 0	Q = 0	Q $\neq$ 0	Q $\neq$ 0

C. Black Hole Entropy

1. *Entropy & Surface Area Relations*

To derive the entropy we need to think of thermodynamic relations. For a certain black hole we can only measure three quantities: mass, total charge, and total angular momentum [1]. Therefore, it would be great if the free energy we are using can be expressed by these “constants”. With this in mind, we will choose Helmholtz free energy F, as T and V all have direct relationship with a black hole’s mass. Thus, they are essentially “held constant”. From the first law of thermodynamics and do a canonical transformation we know:

$$S = \left(\frac{dF}{dT}\right)_{V,N} \tag{4}$$

Now free energy is proportional to mass, and temperature is inversely proportional to mass, carry out the derivation and we get:

$$S \sim GM^2 \sim r_s^2 \sim A \tag{5}$$

Where A is the surface area of the black hole. Surprisingly, entropy is no “extensive” quantity as we normally would expect anymore. This result is first derived 1973 [2].

In reality, entropy has an extra constant term, as mentioned in G Hooft’s paper [3] in 1993. This term is also necessary regarding some “paradoxes”, and is more important than it seems at first glance. This term is, in Hooft’s own words, “The constant C is not known (in fact there could be as yet unknown subdominant terms in this expression, increasing slower than M^2 as the mass increases)." This allows us great freedom in choosing this term, as long as it is bounded and less than infinity. We shall use this to our advantage later.

2. Black Hole Entropy "Capacity"

For those of you (mainly me) who are worried what if the growth of black hole entropy capacity is less than the entropy it received when, for example, a black hole engulfs a star, here is some consolation that black hole does have enough capacity for that additional entropy:

Suppose the initial radius of a black hole is r_s, and final radius after absorbing the star is r'_s. Let parameters of the initial black hole be unprimed. We can write the change in mass and entropy as follows:

$$\Delta M = M' - M \tag{6}$$

$$\Delta S \sim A' - A \sim r_s'^2 - r_s^2 \sim M'^2 - M^2 \tag{7}$$

$$\sim (M' + M)\Delta M \tag{8}$$

Here r_s is proportional to M because Schwarzschild radius is proportional to mass of the black hole. I omitted the gravitational constant and speed of light. Therefore, we can see that whenever some mass falls in, the increase of entropy is always some huge number times the increased mass. Again, surprisingly, although the entropy is not proportional to volume, it still satisfies our requirement for entropy in thermodynamics.

Interestingly, if we let $M = 0$, which means that we cram some infinitesimal mass into a black hole, this will give us:

$$S \sim M'^2 \tag{9}$$

Once again we recovered the entropy-mass relation. This also implies that there is always enough entropy "storage space" for black hole, even for infinitesimally small black holes.

In G Hooft's paper [3], he presented another argument which ultimately lead to a similar result:

$$S_{max} = \frac{A}{4} \tag{10}$$

From this we can safely say that black holes always has the "capacity" to store the entropy as required by second law of thermodynamics.

3. Merging Black Holes

Hawking also reasoned that when two black holes merge, their combined entropy will necessarily increase. It is obvious to see this is true:

$$S_{initial} \sim G(M_1^2 + M_2^2) < G(M_1 + M_2)^2 \sim S_{final} \tag{11}$$

Now this result is known as the second law of black hole mechanics. Incidentally, these many similarities between black hole entropy and thermodynamics entropy is what motivate physicists to explain black hole entropy rather than modifying existing theories, in this case, second law of thermodynamics which states that entropy cannot be destroyed.

III. MATTER IN BOXES

Instead of performing rigorous impeccable calculation, we are going to use the fly by night approach. Moreover, since no one really understands what happens inside a black hole, (and we never would observe anything from inside the event horizon, that is for sure), we would take an indirect approach by using the following argument:

Suppose we have a rigid box full of some particles. Imagine we put more particles of the same kind in, one by one, until the substances collapse into a black hole. Therefore, we could argue that, the behavior of the system is as close a system we understand well can get to the black hole as possible.

It is widely assumed that when something is pressed into its Schwarzschild radius, it will turn into a black hole. Although this might not be strictly correct, we can perhaps say this phenomenon is true on that order of magnitude. We are going to follow the same assumption in the later arguments of specific particles.

A. Photon Gases

Consider a bunch of photons inside a box with length r_s. After they collapse into a black hole, some of their energy will have turn into mass, because black hole with infinitesimally small mass has infinite temperature and will explode instantly according to the temperature-mass relationship mentioned above. Alternatively, we could argue that, even if the photons collapse into a black hole, the black hole evaporates very quickly provided my box is about the size of a lunchbox. In fact, inside the box it is like an equilibrium: photons continuously collapse into black holes, which evaporate in the next instant. With that in mind, we can loosely write:

$$E \sim Mc^2 \sim VT^4 \sim r_s^3 T^4 \quad (12)$$

If we replace r_s by M, we will get:

$$T \sim \frac{1}{\sqrt{M}} \quad (13)$$

It is okay that we did not get the equation in (3), because we are not really solving for complicated differential equations from general relativity. As a sanity check we know this behavior is normal for bosons: when the density of particles is too large, bosons will be forced into Bose-Einstein condensate and thus have a lower temperature.

B. Fermion Gases

Fermion gases is slightly more complicated because we cannot get away without thinking about the phase space. Our box is essentially an infinite square well in three dimensions with length r_s, therefore the wavefunctions are quantized in three dimensions. Namely:

$$\psi_n = sin\left(\frac{n\pi x_i}{r_s}\right) \quad (14)$$

$$n \in \mathbb{Z} \quad (15)$$

As a result, each point in the "n-space" represent a unique particle state. If we draw out all the states, it will be approximately one eighth of a sphere. The number of states in this volume should be N, the total number of particles:

$$N \sim n_{max}^3 \quad (16)$$

Where factor of $\frac{3\pi}{4}$, $\frac{1}{8}$ and 2 are all omitted since we are only interested in proportion relationships. The state just on the boundary of the sphere should have the largest energy, which is usually called fermi energy under zero kelvins. Therefore:

$$\epsilon_F \sim T_F \sim n_{max}^2 \quad (17)$$

Where ϵ_F denotes the fermi energy. Now if we replace the n_{max} with N, the number of particles, we get:

$$T \sim N^{\frac{2}{3}} \sim M^{\frac{2}{3}} \quad (18)$$

Because total mass is just number of particles times individual mass.

Now we get something interesting. If we were to make a black hole out of fermions, it would have incredibly high temperature. This also makes sense because fermions do not like to be near each other, so each additional fermion has to take up a vacant, usually more energetic, state in the system. As we add in more and more fermions, the contents will get hotter and hotter.

This result also implies that we cannot make a black hole out of fermions, at least not the current ones, as they have close to zero kelvins temperature. Actually the universe is way ahead of us. One example of this would be neutron stars, and they usually have a surface temperature of around 600,000 K, albeit they will gradually cool down over time.

C. Cats

Taking the result as granted from Google, the current estimated cat population is about 600 million, and the average mass of a cat is 3.7 kg. Now suppose we ask Mr.Schrödinger to exercise his "cruelty" on cats once more. He would need to cram all these cats into a volume of roughly 100 atoms in order to eliminate the Schrödinger's cat "paradox" once and for all. After all, Einstein spend large quantities of his later times opposing quantum mechanics, and the Copenhagen interpretation in this case.

IV. PARADOXES

Next We want to talk about some paradoxes. There have been many paradoxes in our history of striving to gain an understanding towards the black hole, and we still have many open questions today. Perhaps these questions will be solved if we finally manage to formulate quantum gravity, but to ponder about them may also provide insight or hints of quantum gravity.

A. Information Paradox

According to the no-hiding theorem, information is never lost. This can be rigorously proven and is a fundamental consequence of the linearity and the unitarity of quantum mechanics. Therefore, black boles necessarily need to store all the information of objects it devoured. However, as Hawking's radiation really results from vacuum fluctuations, they seem to carry no information. What makes matter worse is that black holes do have a finite lifetime, albeit very very long. If black holes will exist forever, then we do not necessarily need to worry about this and can just say that black holes store this information somewhere. But if black holes will evaporate some day, we are forced to come to the conclusion that information is lost in black holes.

One possible solution lies in the mechanism of Hawking radiation. We know that entropy is stored on black hole's surface, so probably the engulfed particle pair entangled with the particles on the surface, thus transporting its state to its previously entangled partner, which is escaping the black hole. This process is known as quantum teleportation, and from this point of view we can argue that the information is not lost, but rather excessively encoded by the black hole.

B. Weiheng "Paradox"

Imagine we have two identical cups of tea, except one is very hot, and the other is very cold. Now suppose they are cinnamon flavored tea and we hate them, so we want to throw them in two identical black holes. Because the increase in mass is the same for the two black holes, their final radius should also be the same. Therefore, we face a paradox: obviously the hot tea has more entropy than a cold cup of tea, but after they are absorbed into two black holes it seems that the final entropy of black holes are the same, as the black holes have the same final radius. How could this be?

If you want, you may think of two boxes of otherwise identical cold and hot gas. If doctor strange suddenly modified G to be infinity, then all the molecules will fall down to the ground surface. Ordinarily gravity force is small compared to other basic forces, but with extreme mass like black hole gravity becomes prominent. Now our system is reduced to two dimensional: since all the particle can only exist on the ground! The chaotic 3D motion of particles is reduced to 2D, and guess what, this is precisely the surface of the black hole in this case! It is like the water molecules's motion on the water surface on a ring dipped in a bubble mix, which you can see with your eyes by adjusting the angle under sunlight. On the black hole's surface, there also must be some sort of Brownian motion on the surface and this corresponds to the constant term we talked above in black hole's entropy equation. Therefore, though hot cup of tea and cold cup of tea adds the same mass, their contribution to the constant term may be different!

But wait, we probably should not get too excited yet, as there may exist another possibility. You see, it may turn out that black holes will just turn this extra entropy into extra mass. In this way, its surface area increased a little, and is also able to hold more entropy. But what if the extra entropy corresponds to no particle's mass? Fortunately, conservation of energy does not hold for microscopic levels, and the black hole will just convert it to some nearby particle mass with some probability.

Of course, these are just speculations of the thought experiment. They could well be wrong, and I would need more math to fully solve this problem.

V. EXTRACTING ENERGY FROM BLACK HOLES

In the very distant future, after all of the stars have died, the living beings alive then, if any, cannot acquire their energy as we do today. Currently, most of our energy is directly or indirectly from the sun: animals in the wild feed on plants, which rely on the sun to synthesize organic compounds. Most of our fossil fuel came from photosynthesis of plants millions of years ago. But if all the stars have died, and all the dwarfs cooled down, how could life possibly survive? In the darkest night, with slim hope to thrive, black holes will ignite, the last ray of light.

A. Penrose Process

Now in order to extract energy from the black hole we cannot think merely of Schwarzschild black holes anymore. In fact, most black holes are essentially Kerr black holes, with very little net charge that can be ignored. When supermassive objects rotate, they have a relativistic effect on nearby objects. Lense and Thirring [4] derived this under the framework of general relativity in 1918. In the case of black hole, this results in a larger donut shaped region out of the event horizon called ergosphere, where it is impossible to stand still. Objects that enter the ergosphere can still return, but they are forced to move as you need speed more than c to stand still. Inside ergosphere, objects are allowed to have negative energy as measured by an observer infinitely far away. Imagine an object splits into two after entering the ergosphere, if the part that falls in has negative energy, then the part that escapes will have more energy than initial energy. Therefore, if a spaceship fly by ergosphere, it will experience something very similar to the sling shot effect, only instead of planet's momentum doing work you have the black hole's angular momentum doing work. Indeed, this should really be named as the title of this article, "fly by black hole". This process of extracting energy is called the Penrose process [5], first proposed in 1971.

B. Blandford–Znajek process

Actually, I think figure 2 from Introduction of Black Hole Physics [6] illustrate this idea very well:

The black hole is similar to a conducting body. When you have a spinning body in an external magnetic field, this is essentially a generator. This concept is first proposed by R. D. Blandford and R. L. Znajek in 1977 [7]. They asked if black holes can produce electromagnetic waves like a pulsar, and turns out the answer is yes.

Now we want to reason the potential difference between the pole and equator of a spinning black hole. Suppose the materials outside the event horizon creates a uniform magnetic field B experienced by the black hole. From a very basic equation of rod passing through magnetic field $\Delta V = vBl$, we know magnetic field has dimension of voltage per length times speed. Now, recall that angular momentum has units of length times mass times speed, we can write out the equation since the only relevant mass is the black hole's mass, M:

$$[B] = \frac{V}{v * L} \tag{19}$$

$$[J] = v * M * L \tag{20}$$

$$\Delta V \sim \frac{JB}{M} \tag{21}$$

This indeed agrees with equation (2.2) in Blandford's paper.

VI. CONCLUSION

We have covered most aspects of a black hole: its radius, temperature, entropy, some paradoxes and how we can extract their energy. I want to end with the point that rotational energy of black holes may well be the last energy source for life in the universe. When all the stars are dead, and all the red dwarfs have cooled down and white dwarfs turned into black dwarfs, the universe will become dark for all eternity. "The last living being in

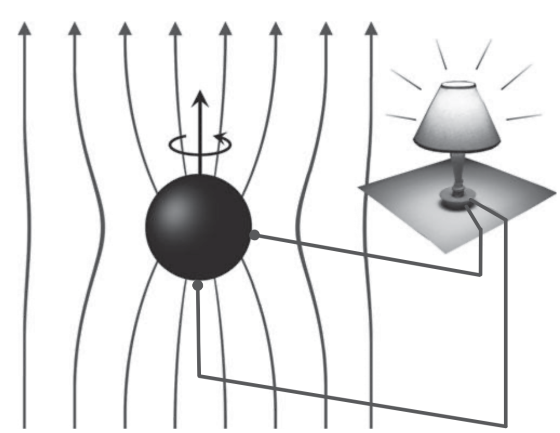

FIG. 2. Black hole as a unipolar inductor. Rotation of a black hole in an external magnetic field generates the electric potential difference between the pole and the equator.[6]

existence might one day end its life around a black hole, which is equally chilling and uplifting. It turns out, even without any light, there are places we can go."[8]

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